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Solutions Of The Equation Of Helmholtz In An Angle 1

Introduction, the F Problem.

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MATHEMATICS

SOLUTIONS OF THE EQUATION OF HELMHOLTZ IN AN ANGLE *). I.

BY

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1. *Introduction*

This is the first of a set of four papers dealing with the problem solved by VAN DANTZIG in a previous publication [1]). The problem is as follows. To determine a function of Green $G(r, \varphi, r_0, \varphi_0)$ in the angle A which in polar coordinates r, φ is given by $\varphi_1 < \varphi < \varphi_2$ satisfying the Helmholtz equation

$$(1.1) \qquad \left(r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \varphi^2} - r^2\right) G(r, \varphi, r_0, \varphi_0) = -r_0 \delta(r - r_0)\, \delta(\varphi - \varphi_0),$$

and the boundary conditions

$$(1.2) \qquad \cos \gamma_j \frac{\partial G}{\partial \varphi} - r \sin \gamma_j \frac{\partial G}{\partial r} = 0 \qquad \text{at } \varphi = \varphi_j$$

for $j = 1$ and $j = 2$.
The constants γ_j may be complex and it will be assumed that

$$(1.3) \qquad -\tfrac{1}{2}\pi < \operatorname{Re} \gamma_j \leqq \tfrac{1}{2}\pi$$

We note that $r_0^{-1}\, \delta(r - r_0)\, \delta(\varphi - \varphi_0)$ represents a two-dimensional point source of unit strength written as a distribution. In Cartesian coordinates x, y we should have written $\delta(x - x_0)\, \delta(y - y_0)$. We believe that by the use of the formalism of the distributions many well-known properties concerning functions of Green can be expressed in a short and elegant way.

The behaviour of the function of Green satisfying (1.1) at r_0, φ_0 is as follows

$$(1.4) \qquad G = -(2\pi)^{-1} \ln \{r^2 + r_0^2 - 2 r r_0 \cos(\varphi - \varphi_0)\}^{\frac{1}{2}} + 0(1).$$

The problem of Green is an essentially non-homogeneous problem. We shall also consider the corresponding homogeneous problem where the right-hand side of (1.1) is replaced by zero. There a function $F(r, \varphi)$ is sought satisfying the Helmholtz equation

$$(1.5) \qquad (\Delta_{r,\varphi} - 1)\, F(r, \varphi) = 0$$

and the boundary conditions (1.2).

*) Report TW 61 of the Mathematical Centre, Amsterdam, Netherlands.
[1]) Cf. D. VAN DANTZIG (1958).

For brevity the problem of Green and the corresponding homogeneous problem will be referred to as the G-problem and the F-problem respectively.

In the first paper we shall consider the F-problem and the G-problem for the potential equation with the boundary conditions (1.2). Next we shall treat the special case where F and G satisfy a Helmholtz equation with the boundary conditions $\partial G/\partial\varphi = 0$ at $\varphi = \varphi_j$. Finally the F-problem in the general case will be considered.

In the second paper we shall treat the G-problem in the general case. Two ways of solution will be presented each having its own merits The first way is a condensed version of Van Dantzig's method who has solved this problem in the paper cited above. The second way is an original one which has been obtained almost at the same time as Van Dantzig's solution.

In the third paper the half-plane case will be studied. Although the solution can be obtained by specialization from the solution of the general case a simple and straight-forward method of solution will be given which rests essentially upon a reduction to a problem of the Wiener-Hopf type.

In the fourth paper a generalization of the G- and F-problem will be considered. There a term with G or F will be added to the boundary conditions (1.2).

The physical applications include the three-dimensional progressive waves on a sloping beach [2]) and the long waves in a rotating angular bay due to a windfield. In this and the following papers we shall use the notation by VAN DANTZIG in his paper with some minor modifications. In particular we shall write $\theta = \varphi_2 - \varphi_1$, and $\nu = \pi/\theta$.

2. *A potential problem*

In this section the problem of the preceding section will be solved for Laplace's equation instead of Helmholtz' equation. The F-problem will be considered first. We have

$$\Delta F = 0 \tag{2.1}$$

$$\cos\gamma_j \frac{\partial F}{\partial\varphi} - r \sin\gamma_j \frac{\partial F}{\partial r} = 0 \quad \text{at } \varphi = \varphi_j \tag{2.2}$$

for $j = 1$ and $j = 2$.

Let us take the trial solution

$$F = A\,(re^{i\varphi})^\lambda + B(re^{-i\varphi})^\lambda, \tag{2.3}$$

where A, B and λ are constants.

Substitution of (2.3) in (2.2) gives for $j = 1, 2$

$$\lambda\{A \exp i(\gamma_j + \lambda\varphi_j) - B \exp -i(\gamma_j + \lambda\varphi_j)\} = 0. \tag{2.4}$$

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[2]) Cf. A. S. PETERS (1952) and H. A. LAUWERIER (1959).

Elimination of A and B gives either $\lambda=0$ or

$$\theta\lambda=\gamma_1-\gamma_2+m\pi, \tag{2.5}$$

where m is an integer. From (2.3) we obtain the solution

$$F(r,\varphi)=r^{\lambda}\cos\{\lambda(\varphi-\varphi_1)-\gamma_1\}. \tag{2.6}$$

From (2.5) and (2.6) a set of solutions is obtained which, with the exception, of course, of $F=$ constant are infinite either at $r=\infty$ or at $r=0$.

In the special case $\gamma_1=\gamma_2=\gamma$ a second solution can be obtained by formal differentiation of (2.6) with respect to λ and letting $\lambda\to 0$. In this way we obtain the set of solutions

$$1,\ (\cos\gamma)\ln r+\varphi\sin\gamma,\ r^{m\nu}\cos\{m\nu(\varphi-\varphi_1)-\gamma\} \tag{2.7}$$

where $m=\pm 1,\ \pm 2,\ \ldots.$

We next consider the G-problem. A particular solution may be derived systematically in the following way. Let us assume for the moment that γ_1 and γ_2 are real. Then G can be considered as the real part of an analytic function

$$G=\operatorname{Re}\phi(z), \tag{2.8}$$

where $z=re^{i\varphi}$. By means of the conformal transformation $e^{-i\nu\varphi_1}z^{\nu}\to z$ the sector $\varphi_1<\varphi<\varphi_2$ can be mapped upon the upper halfplane $0<\varphi<\pi$. There the problem reduces to that of finding an analytic function $\phi(z)$ for which

$$\operatorname{Im}\left(e^{i\gamma_1}\frac{d\phi}{dz}\right)=0 \quad \text{at } y=0,\ x>0, \tag{2.9}$$

$$\operatorname{Im}\left(e^{i\gamma_2}\frac{d\phi}{dz}\right)=0 \quad \text{at } y=0,\ x<0, \tag{2.10}$$

and for which at $z=z_0$

$$\phi(z)=-(2\pi)^{-1}\ln(z-z_0)+0(1). \tag{2.11}$$

This problem can be solved by means of the incomplete Beta-function

$$\Psi(z,\mu)=\int_0^z\frac{t^{\mu-1}}{1-t}\,dt, \tag{2.12}$$

where $\operatorname{Re}\mu>0$. Some properties of this function follow below

$$\Psi(z,\mu)=\mu^{-1}z^{\mu}F(\mu,1;\mu+1;z), \tag{2.13}$$

$$\Psi(z,\mu)=\sum_{k=0}^{\infty}\frac{z^{\mu+k}}{\mu+k},\quad |z|<1. \tag{2.14}$$

If $0<\operatorname{Re}\mu<1$ we have

$$\Psi(z,\mu)=\Psi(z^{-1},1-\mu)+\pi\operatorname{cosec}\mu\pi\cdot\exp\{i\mu\pi\operatorname{sgn}(\arg z)\}, \tag{2.15}$$

and

$$\Psi(z,\mu)=\sum_{k=1}^{\infty}\frac{z^{\mu-k}}{k-\mu}+\pi\operatorname{cosec}\mu\pi\cdot\exp\{i\mu\pi\operatorname{sgn}(\arg z)\} \tag{2.16}$$

for $|z|>1$.

If in the first place $\gamma_1=\gamma_2=\gamma$ we see that by taking

$$\frac{d\phi}{dz}=-\frac{e^{-i\gamma}}{2\pi}\left\{\frac{e^{i\gamma}}{z-z_0}+\frac{e^{-i\gamma}}{z-\bar{z}_0}\right\}$$

a solution of (2.9) and (2.10) is obtained which also satisfies (2.11). By integration we obtain the particular solution

$$\phi(z)=-\frac{1}{2\pi}\left\{\ln\left(1-\frac{z}{z_0}\right)+e^{-2i\gamma}\ln\left(1-\frac{z}{\bar{z}_0}\right)\right\}. \tag{2.17}$$

The Green's function following from this is finite at $r=0$ but diverges as $\ln r$ at infinity. The solutions (2.7) of the F-problem show that in this case there is no Green's function which is bounded both at $r=0$ and $r=\infty$. The explicit form of the Green's function is easily obtained by applying the inverse transformation $z\rightarrow z^{\nu}\exp\text{-}i\nu\varphi_1$ and by taking the real part viz.

$$\begin{cases} G(r,\varphi,r_0,\varphi_0)=-\frac{1}{4\pi}\left\{\ln\left(1-\frac{r^{\nu}}{r_0^{\nu}}\exp i\nu(\varphi-\varphi_0)\right)+\right. \\ +\ln\left(1-\frac{r^{\nu}}{r_0^{\nu}}\exp -i\nu(\varphi-\varphi_0)\right)+e^{-2i\gamma}\ln\left(1-\frac{r^{\nu}}{r_0^{\nu}}\exp i\nu(\varphi+\varphi_0-2\varphi_1)\right)+ \\ \left.+e^{2i\gamma}\ln\left(1-\frac{r^{\nu}}{r_0^{\nu}}\exp -i\nu(\varphi+\varphi_0-2\varphi_1)\right)\right\}. \end{cases} \tag{2.18}$$

If in the second place $\gamma_1\neq\gamma_2$ we see that by taking

$$\frac{d\phi}{dz}=-\frac{1}{2\pi}\left\{\frac{(z/z_0)^{\lambda\nu}}{z-z_0}+e^{-2i\gamma_1}\frac{(z/\bar{z}_0)^{\lambda\nu}}{z-\bar{z}_0}\right\}, \tag{2.19}$$

where λ is given by (2.5) a solution of (2.9), (2.10) and (2.11) is obtained. It is possible to choose the integer m in (2.5) in such a way that the resulting Green's function is bounded at $r=0$ and $r=\infty$. This special choice will be made by defining the constant μ in the following way

$$\begin{cases} \mu=\pi^{-1}(\gamma_1-\gamma_2) \text{ for } \gamma_1>\gamma_2, \\ \mu=1-\pi^{-1}(\gamma_2-\gamma_1) \text{ for } \gamma_1<\gamma_2. \end{cases} \tag{2.20}$$

Then it follows from (2.19) by integration that we may take

$$2\pi\phi(z)=\Psi\left(\frac{z}{z_0},\mu\right)+e^{-2i\gamma_1}\Psi\left(\frac{z}{\bar{z}_0},\mu\right). \tag{2.21}$$

The explicit form of the Green's function is

$$\begin{cases} G(r,\varphi,r_0,\varphi_0)=\pi\Psi\left(\frac{r^{\nu}}{r_0^{\nu}}\exp i\nu(\varphi-\varphi_0),\mu\right)+ \\ +\pi\Psi\left(\frac{r^{\nu}}{r_0^{\nu}}\exp\text{-}i\nu(\varphi-\varphi_0),\mu\right)+\pi e^{-2i\gamma_1}\Psi\left(\frac{r^{\nu}}{r_0^{\nu}}\exp i\nu(\varphi+\varphi_0-2\varphi_1),\mu\right)+ \\ +\pi e^{2i\gamma_1}\Psi\left(\frac{r^{\nu}}{r_0^{\nu}}\exp\text{-}i\nu(\varphi+\varphi_0-2\varphi_1),\mu\right). \end{cases} \tag{2.22}$$

Since $0<\mu<1$ this Green's function vanishes at $r=0$ and has a limit at $r\rightarrow\infty$. From (2.16) it follows that

$$\begin{cases} \lim\limits_{r\rightarrow\infty} G=\frac{1}{2}\operatorname{cosec}\mu\pi\{\cos\mu\pi+\cos(2\gamma_1-\mu\pi)\}= \\ \qquad=\cos\gamma_1\cos\gamma_2\operatorname{cosec}(\gamma_1-\gamma_2). \end{cases} \tag{2.23}$$

If this limit is subtracted from the right-hand side of (2.22) a Green's function is obtained which vanishes at $r=\infty$ and has a limit at $r \to 0$. From (2.14) and (2.16) expansions of G can be derived in the neighbourhood of $r=0$ or for $r \to \infty$.

There is no difficulty in extending the results found above for complex values of γ_1 and γ_2.

3. *The case* $\gamma_1=\gamma_2=0$

The F-problem reduces in this case to

$$(3.1) \qquad (\Delta-1)F=0$$

$$(3.2) \qquad \frac{\partial F}{\partial \varphi}=0 \qquad \text{at } \varphi=\varphi_j$$

for $j=1$ and $j=2$.

The F-functions can be found by separating the variables in (3.1). Without difficulty the following set is obtained

$$(3.3) \qquad \begin{cases} F_m(r,\varphi)=\mathrm{K}_{m\nu}(r)\cos m\nu\,(\varphi-\varphi_1), \\ F^*_m(r,\varphi)=\mathrm{I}_{m\nu}(r)\cos m\nu\,(\varphi-\varphi_1), \end{cases}$$

where $m=0, 1, 2, \ldots$

Next we consider the G-problem. In the full plane where boundary conditions are absent Green's standard function is

$$(3.4) \qquad G(r,\varphi,r_0,\varphi_0)=\frac{1}{2\pi}\mathrm{K}_0\left(\sqrt{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)}\right).$$

For the moment this function will be written in the abbreviated notation $\mathrm{K}(\varphi_0)$. Then by applying the well-known principle of reflection Green's function in the case $\theta=\pi/m$ where m is a positive integer, may be written as

$$(3.5) \qquad G(r,\varphi,r_0,\varphi_0)=\sum_{j=0}^{m-1}\{\mathrm{K}(\varphi_0+2j\pi/m)+\mathrm{K}(-\varphi_0+2j\pi/m)\}.$$

For arbitrary values of θ the following result has been found

$$(3.6) \qquad G(r,\varphi,r_0,\varphi_0)=\begin{cases} 2\theta^{-1}\sum\limits_{m=0}^{\infty}{}^{*}\,\mathrm{K}_{m\nu}(r)\,\mathrm{I}_{m\nu}(r_0)\cos m\nu\,(\varphi-\varphi_1)\cos m\nu(\varphi_0-\varphi_1) & \text{for } r>r_0 \\ 2\theta^{-1}\sum\limits_{m=0}^{\infty}{}^{*}\,\mathrm{I}_{m\nu}(r)\,\mathrm{K}_{m\nu}(r_0)\cos m\nu(\varphi-\varphi_1)\cos m\nu(\varphi_0-\varphi_1) & \text{for } r<r_0. \end{cases}$$

The asterisk of the summation sign indicates that to the term with $m=0$ a factor $\frac{1}{2}$ has to be added.

We shall verify (3.6) by direct substitution in (3.1) by making use of the formalism of the theory of distributions. If the symbol $\iota(x)$ denotes the function which vanishes for negative values of the argument and equals 1 for $x \geqq 0$, we write

$$G=2\theta^{-1}\sum_{m=0}^{\infty}{}^{*}\{\mathrm{K}_{m}(r)\,\mathrm{I}_{m\nu}(r_0)\,\iota(r-r_0)+\mathrm{I}_{m\nu}(r)\,\mathrm{K}_{m\nu}(r_0)\,\iota(r_0-r)\}\cdot \\ \cdot\cos m\nu\,(\varphi-\varphi_1)\cos m\nu\,(\varphi_0-\varphi_1).$$

By taking weak derivatives we obtain indeed

$$(\Delta-1)\,G = 2\theta^{-1}\sum_{m=0}^{\infty}{}^{*}\,\{\mathrm{K}'_{m\nu}(r_0)\,\mathrm{I}_{m\nu}(r_0)-\mathrm{I}'_{m\nu}(r_0)\,\mathrm{K}_{m\nu}(r_0)\}\,\delta(r-r_0)\cdot$$
$$\cdot\cos m\nu\,(\varphi-\varphi_1)\cos m\nu\,(\varphi_0-\varphi_1) =$$
$$= -r_0^{-1}\,\delta(r-r_0)\cdot 2\theta^{-1}\sum_{m=0}^{\infty}{}^{*}\cos m\nu\,(\varphi-\varphi_1)\cos m\,\nu(\varphi_0-\varphi_1) =$$
$$= -r_0^{-1}\,\delta(r-r_0)\,\delta(\varphi-\varphi_0).$$

A systematic but rather lengthy derivation of the result (3.6) will be given in the second paper of this series.

If in particular we take $\varphi_0=\varphi_1=0$ and $\varphi_2=\pi$ from (3.5) and (3.6) two equivalent expressions are obtained giving

$$(3.7)\quad \mathrm{K}_0(\sqrt{r^2+r_0^2-2rr_0\cos\varphi}) = \begin{cases} 2\sum_{m=0}^{\infty}{}^{*}\,\mathrm{K}_m(r)\,\mathrm{I}_m(r_0)\cos m\varphi, & r>r_0,\\ 2\sum_{m=0}^{\infty}{}^{*}\,\mathrm{I}_m(r)\,\mathrm{K}_m(r_0)\cos m\varphi, & r<r_0.\end{cases}$$

This is the well-known addition formula for the modified Bessel functions.

If in (3.6) the cosines are replaced by sines Green's function with the boundary conditions

$$(3.8)\qquad G=0 \quad \text{for} \quad \varphi=\varphi_j$$

is obtained.

Green's function (3.6) is finite at $r=0$ and vanishes exponentially at infinity. We note that at $r=0$

$$(3.9)\qquad G(0,\varphi,r_0,\varphi_0)=\theta^{-1}\mathrm{K}_0(r_0).$$

4. *A functional relation*

In this section we derive some properties of Van Dantzig's auxiliary function $e(z,\gamma)$ [3]). They will be used in the following sections in the treatment of the general case of section 1. This function satisfies the functional relation

$$(4.1)\qquad \frac{e(z+\mathrm{i}\theta,\gamma)}{e(z-\mathrm{i}\theta,\gamma)}=\frac{\mathrm{ch}\,(z+\mathrm{i}\gamma)}{\mathrm{ch}\,(z-\mathrm{i}\gamma)},$$

is even in z, and is normalized by $e(0,\gamma)=1$.

Assuming that $d/dz\ln e(z,\gamma)$ can be represented as a sine transform we put

$$(4.2)\qquad e'(z,\gamma)/e(z,\gamma)=\int_0^{\infty}\sin tz\,\psi(t)\,dt.$$

Logarithmic differentiation of (4.1) gives

$$e'(z+\mathrm{i}\theta,\gamma)/e(z+\mathrm{i}\theta,\gamma)-e'(z-\mathrm{i}\theta,\gamma)/e(z-\mathrm{i}\theta,\gamma)=\mathrm{th}(z+\mathrm{i}\gamma)-\mathrm{th}(z-\mathrm{i}\gamma).$$

[3]) Cf. D. van Dantzig l.c. section 6.

By substitution of (4.2) we obtain

$$2\mathrm{i}\int_0^\infty \cos tz \,\mathrm{sh}\, \theta t\, \psi(t)dt = \mathrm{th}(z+\mathrm{i}\gamma)+\mathrm{th}(-z+\mathrm{i}\gamma). \tag{4.3}$$

The inversion of (4.3) can be performed without difficulty. One finds easily

$$\psi(t) = \frac{\mathrm{sh}\,\gamma t}{\mathrm{sh}\theta t\,\mathrm{sh}\,\frac{1}{2}\pi t} \tag{4.4}$$

so that with the proper normalization it follows from (4.2) and (4.4) that

$$e(z,\gamma) = \exp\left[\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos tz}{t}\,\frac{\mathrm{sh}\,\gamma t}{\mathrm{sh}\theta t\,\mathrm{sh}\,\frac{1}{2}\pi t}\,dt\right]. \tag{4.5}$$

This solution of (4.1) was obtained by VAN DANTZIG in a different way. This expression converges for $|\mathrm{Im}\, z| < \theta + \frac{1}{2}\pi - |\mathrm{Re}\,\gamma|$.

The analytic continuation of $e(z, \gamma)$ can be found by expansion into an infinite product. We shall use the following well-known Laplace transform

$$\int_0^\infty \mathrm{e}^{-pt}\,\frac{1-\cos at}{t}\,dt = \tfrac{1}{2}\ln\left(1+\frac{a^2}{p^2}\right). \tag{4.6}$$

From (4.5) we can derive

$$e(z,\gamma) = \exp\left[2\int_0^\infty \frac{1-\cos tz}{t}\sum_{m,n}\{\mathrm{e}^{-(S-\gamma)t}-\mathrm{e}^{-(S+\gamma)t}\}\,dt\right],$$

where

$$S = (2m+1)\theta + (2n+1)\tfrac{1}{2}\pi,$$

and where m and n run through the non-negative integers. Application of (4.6) gives [4])

$$e(z,\gamma) = \prod_{m,n}\left\{1+\frac{z^2}{(S-\gamma)^2}\right\}\left\{1+\frac{z^2}{(S+\gamma)^2}\right\}^{-1}. \tag{4.7}$$

Hence $e(z, \gamma)$ is a meromorphic function with the following poles and zeros

$$\text{poles}\quad z = \pm\,\mathrm{i}\{(2m+1)\theta+(2n+1)\tfrac{1}{2}\pi+\gamma\} \tag{4.8}$$

$$\text{zeros}\quad z = \pm\,\mathrm{i}\{(2m+1)\theta+(2n+1)\tfrac{1}{2}\pi-\gamma\}. \tag{4.9}$$

Either from (4.5) or from (4.7) we can derive the elementary cases

$$e(z, \tfrac{1}{2}\pi) = \mathrm{ch}\,\tfrac{1}{2}\nu z \tag{4.10}$$

and

$$e(z, \theta) = \mathrm{ch}\, z. \tag{4.11}$$

Furthermore it is obvious that

$$e(z, -\gamma) = \{e(z, \gamma)\}^{-1}. \tag{4.12}$$

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[4]) Ibid. cf. formula (6.11) and (6.17).

From (4.5) or from (4.7) also the following functional relation can be derived

(4.13) $$\frac{e(z+\frac{1}{2}\pi\mathrm{i},\gamma)}{e(z-\frac{1}{2}\pi\mathrm{i},\gamma)}=\frac{\mathrm{ch}\,(\frac{1}{2}\nu(z+\mathrm{i}\gamma))}{\mathrm{ch}\,(\frac{1}{2}\nu(z-\mathrm{i}\gamma))}.$$

In a similar way it can be proved that

(4.14) $$\frac{e(z-\mathrm{i}\theta,\gamma)}{e(z,\gamma-\theta)}=C\,\mathrm{ch}\,(z-\mathrm{i}\gamma),$$

where $C=C(\theta,\gamma)$ is independent of z.

An alternative expression for $e(z,\gamma)$ can be derived in the following way. The inversion of (4.2) may be written as

$$\mathrm{sh}\,\theta t\;\psi(t)=(\pi\mathrm{i})^{-1}\int_0^\infty \cos tu\{\mathrm{th}(u+\mathrm{i}\gamma)+\mathrm{th}(-u+\mathrm{i}\gamma)\}du.$$

If this is substituted in (4.2) we obtain

$$e'(z,\gamma)/e(z,\gamma)=\frac{1}{\pi\mathrm{i}}\int_0^\infty\{\mathrm{th}\,(u+\mathrm{i}\gamma)+\mathrm{th}\,(-u+\mathrm{i}\gamma)\}\,du\int_0^\infty \sin zt\frac{\cos ut}{\mathrm{sh}\,\theta t}\,dt.$$

In view of Erdelyi *et al. Integral transforms I formula* 1.9.53 this becomes

$$e'(z,\gamma)/e(z,\gamma)=\frac{1}{2\theta\mathrm{i}}\int_0^\infty\{\mathrm{th}\,(u+\mathrm{i}\gamma)+\mathrm{th}\,(-u+\mathrm{i}\gamma)\}\frac{\mathrm{sh}\,\nu z}{\mathrm{ch}\,\nu u+\mathrm{ch}\,\nu z}\,du.$$

By partial integration it follows that

$$\ln e(z,\gamma)=\frac{1}{2\pi\mathrm{i}}\int_0^\infty \ln\,(\mathrm{ch}\,\nu u+\mathrm{ch}\,\nu z)\,d\left\{\ln\frac{\mathrm{ch}\,(u+\mathrm{i}\gamma)}{\mathrm{ch}\,(u-\mathrm{i}\gamma)}-2\mathrm{i}\gamma\right\}+\mathit{constant}=$$
$$=\frac{\gamma}{\pi}\ln\,(1+\mathrm{ch}\,\nu z)-\frac{\nu}{2\pi\mathrm{i}}\int_0^\infty \ln\frac{1+\exp-2\,(u+\mathrm{i}\gamma)}{1+\exp-2\,(u-\mathrm{i}\gamma)}\,\frac{\mathrm{sh}\,\nu u}{\mathrm{ch}\,\nu u+\mathrm{ch}\,\nu z}\,du+$$
$$+\,\mathit{constant},$$

so that

(4.16) $$e(z,\gamma)=C\,(1+\mathrm{ch}\,\nu z)^{\gamma/\pi}\exp\left[-\frac{\nu}{2\pi\mathrm{i}}\int_0^\infty \ln\frac{1+\exp-2\,(u+\mathrm{i}\gamma)}{1+\exp-2\,(u-\mathrm{i}\gamma)}\,\frac{\mathrm{sh}\,\nu u}{\mathrm{ch}\,\nu u+\mathrm{ch}\,\nu z}\,du\right],$$

where C is a constant.

From (4.15) one can derive without difficulty that for Re $z\to\infty$ [5])

(4.16) $$\begin{cases}\ln e(z,\gamma)=\pi^{-1}\gamma\ln\mathrm{ch}\,\nu z+C_0+O(\mathrm{e}^{-\nu z})\ \text{for}\ \nu<2,\\ \qquad\qquad\qquad\qquad\qquad\quad +O(\mathrm{e}^{-2z})\ \text{for}\ \nu>2,\\ \qquad\qquad\qquad\qquad\qquad\quad +O(z\mathrm{e}^{-2z})\ \text{for}\ \nu=2.\end{cases}$$

5. *The F-problem*

In this section the F-problem of section 1 will be considered for arbitrary complex values of γ_1 and γ_2. We shall follow Van Dantzig's method [6]) in a somewhat generalized version.

[5]) Ibid. cf. formula (3.10) and (6.19).

[6]) Cf. D. van Dantzig, l.c. section 2.

The results of section 3 suggest that there exists a class of solutions $F(r, \varphi)$ which vanish at infinity at least as rapidly as $\exp(-cr)$ where c is a positive constant.

Let $F(r, \varphi)$ be such a function and let $r=0$ be a possibly singular point. Then we may form its generalized cosine transform [7])

$$\Psi(s, \varphi)=\pi^{-1}\int_0^\infty \cos rs\, F(r, \varphi)\, dr. \tag{5.1}$$

Its inversion is

$$F(r, \varphi)=\int_{-\infty}^{\infty} \cos\, rs\;\; \Psi(s, \varphi)\, ds. \tag{5.2}$$

This formula can be brought in the form

$$F(r, \varphi)=\int_{-\infty}^{\infty} \cos\,(r \operatorname{sh} u)\; U(u, \varphi)\, du, \tag{5.3}$$

where

$$U(u, \varphi)=\operatorname{ch} u\; \Psi(\operatorname{sh} u, \varphi).$$

By partial integration it can be proved that [8])

$$\frac{\partial^2 U}{\partial u^2}+\frac{\partial^2 U}{\partial \varphi^2}=0. \tag{5.4}$$

Hence $U(u, \varphi)$ is a harmonic function and we may put

$$U(u, \varphi)=\tfrac{1}{2}\{f(u+i\varphi)+f(-u+i\varphi)\}, \tag{5.5}$$

where $f(w)$ is an analytic function of its complex argument w. By substitution of (5.5) in (5.3) we obtain

$$F(r, \varphi)=\tfrac{1}{2}\int_{-\infty}^{\infty} \cos\,(r \operatorname{sh} u)\{f(u+i\varphi)+f(-u+i\varphi)\}\, du. \tag{5.6}$$

If inversely in (5.6) the function $f(w)$ is analytic and if for $u \to \pm\infty$ we have $f(u+i\varphi)=0(|u|^c)$ for some real c then the right-hand side of (5.6) is a regular solution of the Helmholtz equation in the angle $\varphi_1 \leqq \varphi \leqq \varphi_2$ with the possible exception of $r=0$.

If upon (5.6) the boundary conditions (1.5) are applied we obtain after elementary reductions for $j=1$ and $j=2$

$$\int_{-\infty}^{\infty} \cos\,(r \operatorname{sh} u)\{\operatorname{ch}(u-i\gamma_j)f(u+i\varphi_j)-\operatorname{ch}(u+i\gamma_j)f(-u+i\varphi_j)\}\, du=0. \tag{5.7}$$

This is true if

$$\operatorname{ch}(u-i\gamma_j)f(u+i\varphi_j)=\operatorname{ch}(u+i\gamma_j)f(-u+i\varphi_j). \tag{5.8}$$

In the special case $\gamma_1=0$, $\gamma_2=\gamma$ the relations (5.8) reduce to the functional equations

$$f(u)=f(-u), \qquad \frac{f(u+i\theta)}{f(u-i\theta)}=\frac{\operatorname{ch}(u+i\gamma)}{\operatorname{ch}(u-i\gamma)}. \tag{5.9}$$

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[7]) Cf. M. J. LIGHTHILL (1958).

[8]) Cf. D. VAN DANTZIG, l.c. Theorem 1.

In the previous section we have seen that the system (5.9) has the particular solution $e(u, \gamma)$ and the general solution $e(u, \gamma)$ ch $m\nu u$ ($m=0, 1, 2, \ldots$). It is now easily seen that (5.8) has the general solution

$$f(u) = \frac{e(u-\mathrm{i}\varphi_1, \gamma_2)}{e(u-\mathrm{i}\varphi_2, \gamma_1)} \operatorname{ch}(m\nu(u-\mathrm{i}\varphi_1)). \tag{5.10}$$

The generalization in comparison with Van Dantzig's paper consists of the factor $\operatorname{ch}(m\nu(u-\mathrm{i}\varphi_1))$ made possible by the fact that we released the conditions at $r=0$ and $r=\infty$.

If the right-hand side of (5.10) is denoted by $\phi_m(u)$ where ϕ_0 and ϕ are equivalent notations, then we have obtained the following class of regular solutions

$$F_m(r, \varphi) = \int_{-\infty}^{\infty} \cos(r \operatorname{sh} u)\phi_m(u+\mathrm{i}\varphi)\, du. \tag{5.11}$$

Since $\phi_m(w)$ is holomorphic in the strip $\varphi_1 \leqq \operatorname{Im} w \leqq \varphi_2$ the solutions F_m are regular with the possible exception of $r=0$. According to the results of the previous section we have for complex w the asymptotic behaviour

$$\ln \phi(w) = \theta^{-1}(\gamma_2-\gamma_1) \ln \operatorname{ch} w + O(1), \quad \operatorname{Re} w \to \pm\infty. \tag{5.12}$$

Hence it follows that for $r \to 0$

$$\begin{cases} \gamma_1 > \gamma_2 & F(r, \varphi) \quad \text{finite} \\ \gamma_1 = \gamma_2 & F(r, \varphi) = C \ln r + O(1), \\ \gamma_1 < \gamma_2 & F(r, \varphi) = C r^{\theta^{-1}(\gamma_1-\gamma_2)}(1+o(1)), \end{cases} \tag{5.13}$$

where C is a constant. For $m \geqq 1$ we have the general result that for $r \to 0$

$$\ln F(r, \varphi) = -\theta^{-1}(\gamma_2-\gamma_1+m\pi) \ln r + O(1). \tag{5.14}$$

By way of illustration we shall consider the special case $\gamma_1=\gamma_2=0$. In this case the solution is well-known viz.

$$F_m(r, \varphi) = \mathrm{K}_{m\nu}(r) \cos(m\nu(\varphi-\varphi_1)).$$

For non-integer $m\nu$ its generalized cosine transform according to (5.1) is

$$\Psi(s, \varphi) = \frac{\sec \frac{1}{2} m\nu\pi}{4\sqrt{s^2+1}} \{(s+\sqrt{s^2+1})^{m\nu} + (s+\sqrt{s^2+1})^{-m\nu}\} \cos(m\nu(\varphi-\varphi_1)).$$

Further by putting $s = \operatorname{sh} u$

$$U(u, \varphi) = (2\pi)^{-1} \sec \tfrac{1}{2} m\nu\pi \operatorname{ch} m\nu u \cos(m\nu(\varphi-\varphi_1))$$

and

$$f(w) = (2\pi)^{-1} \sec \tfrac{1}{2} m\nu\pi \operatorname{ch}(m\nu(w-\mathrm{i}\varphi_1)).$$

On the other hand we obtain from (5.10) the equivalent result $f(w) = \operatorname{ch}(m\nu(w-\mathrm{i}\varphi_1))$ differing from the above obtained result by a constant factor only.

We shall now consider the solution (5.11) for $m=0$ in more detail.

From (4.8) and (4.9) it follows that $\phi(w)$ has the following poles

$$w = \mathrm{i}\{\varphi_1 \pm (m\theta + \tfrac{1}{2}\pi n + \gamma_2)\}$$
$$w = \mathrm{i}\{\varphi_2 \pm (m\theta + \tfrac{1}{2}\pi n - \gamma_1)\},$$

and zeros

$$w = \mathrm{i}\{\varphi_1 \pm (m\theta + \tfrac{1}{2}\pi n - \gamma_2)\}$$
$$w = \mathrm{i}\{\varphi_2 \pm (m\theta + \tfrac{1}{2}\pi n + \gamma_1)\},$$

where m and n are odd natural numbers.

Hence the representation (5.11) gives not only the solution in the angle $\varphi_1 \leqq \varphi \leqq \varphi_2$ but also its continuation in the wider angle

$$\max(\varphi_1 - \tfrac{1}{2}\pi + \gamma_1, 2\varphi_1 - \varphi_2 - \tfrac{1}{2}\pi - \gamma_2) < \varphi < \min(\varphi_2 + \tfrac{1}{2}\pi + \gamma_2, 2\varphi_2 - \varphi_1 + \tfrac{1}{2}\pi - \gamma_1).$$

In some cases the solution exists in a still wider angle. We note the trivial case $\gamma_1 = \gamma_2 = 0$ with $\phi(w) \equiv 1$. Another case is the following one

$$\gamma_1 + \varphi_1 = \gamma_2 + \varphi_2 = \gamma$$

In this case $\phi(w)$ reduces to an elementary expression since now a set of poles is neutralized by corresponding zeros. According to (4.14) $\phi(w)$ is equivalent to

$$\phi(w) = \operatorname{sech}(w - \mathrm{i}\gamma). \tag{5.15}$$

By substitution of this expression in (5.11) it can easily be shown by means of the calculus of residues that

$$F(r, \varphi) = \pi \exp(-r\,|\cos(\varphi - \gamma)|). \tag{5.16}$$

Let us assume for the moment that γ is real. Then (5.16) gives the required solution in the angle $\varphi_1 \leqq \varphi \leqq \varphi_2$ and its analytic continuation in the halfplane $\gamma - \tfrac{1}{2}\pi < \varphi < \gamma + \tfrac{1}{2}\pi$. At the boundaries $\varphi = \gamma \pm \tfrac{1}{2}\pi$ the factor $\cos(\varphi - \gamma)$ changes sign so that analytic continuation across these boundaries gives a solution which is no longer bounded at infinity. Formula (5.16) gives in virtue of the absolute sign a non-analytic continuation i.e. a solution with a discontinuous derivative at $\varphi = \gamma \pm \tfrac{1}{2}\pi$. The state of affairs is illustrated below in figure 1.

In the general case with arbitrary θ, γ_1 and γ_2 we have a similar behaviour Formula (5.11) with $m = 0$ gives the required solution in an angle which is determined by the "nearest" poles of $\phi(w)$. Analytic continuation outside this angle yields a solution which does not vanish at infinity. However, formula (5.11) gives a non-analytic continuation for all φ values for which the integrand has no poles. At the lines $\varphi = \varphi_p$ for which there are poles the solution has a discontinuous derivative. These lines can be interpreted as lines of logarithmic sources.

A typical case with real γ_1 and γ_2 is illustrated in figure 2.

We next consider the important case where θ, γ_1 and γ_2 are subjected to the following inequalities

$$\tfrac{1}{2}\pi < \theta < \pi, \quad \operatorname{Re}\gamma_1 < \theta - \tfrac{1}{2}\pi, \quad \operatorname{Re}\gamma_2 > \tfrac{1}{2}\pi - \theta. \tag{5.17}$$

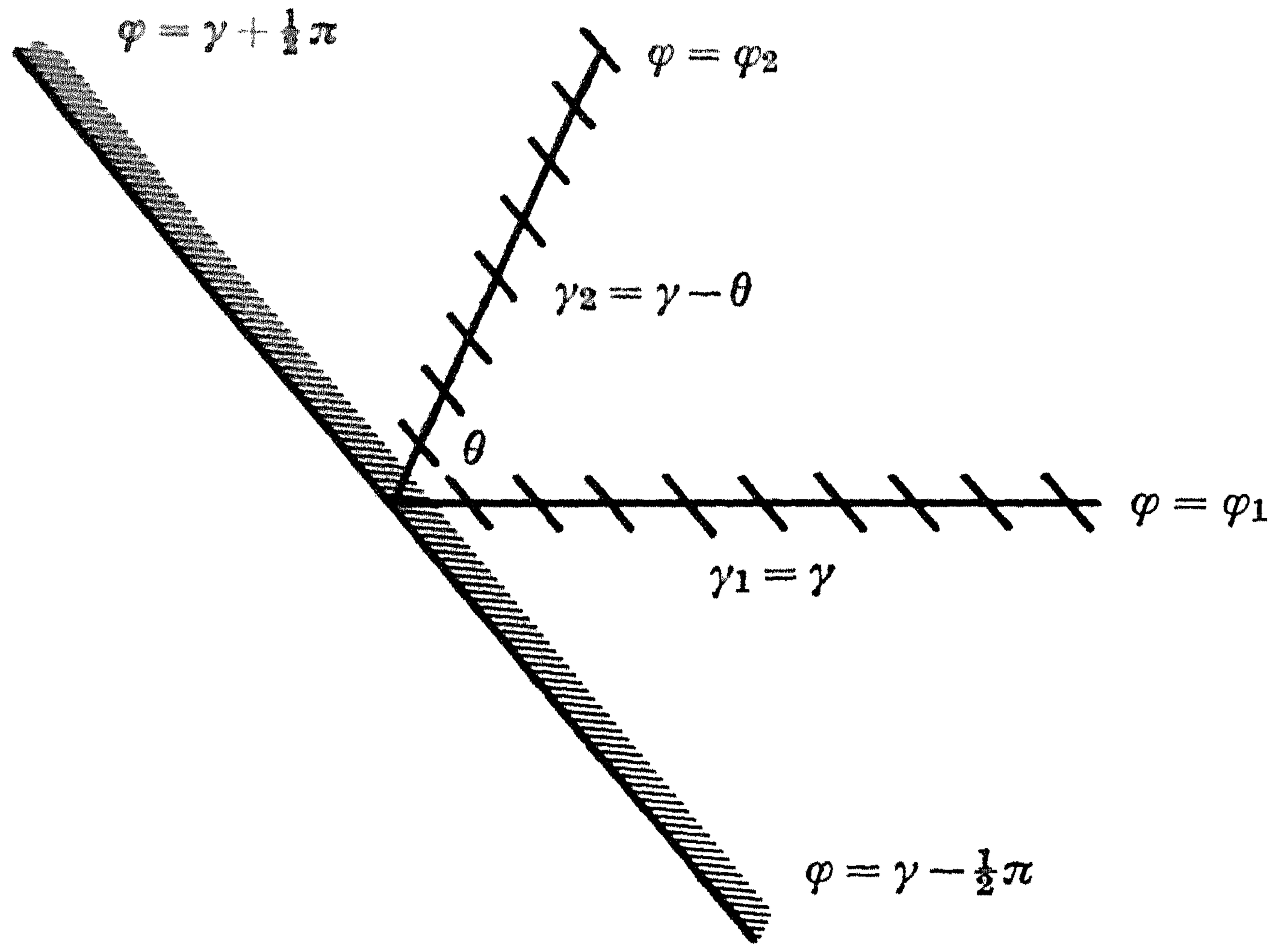

Fig. 1

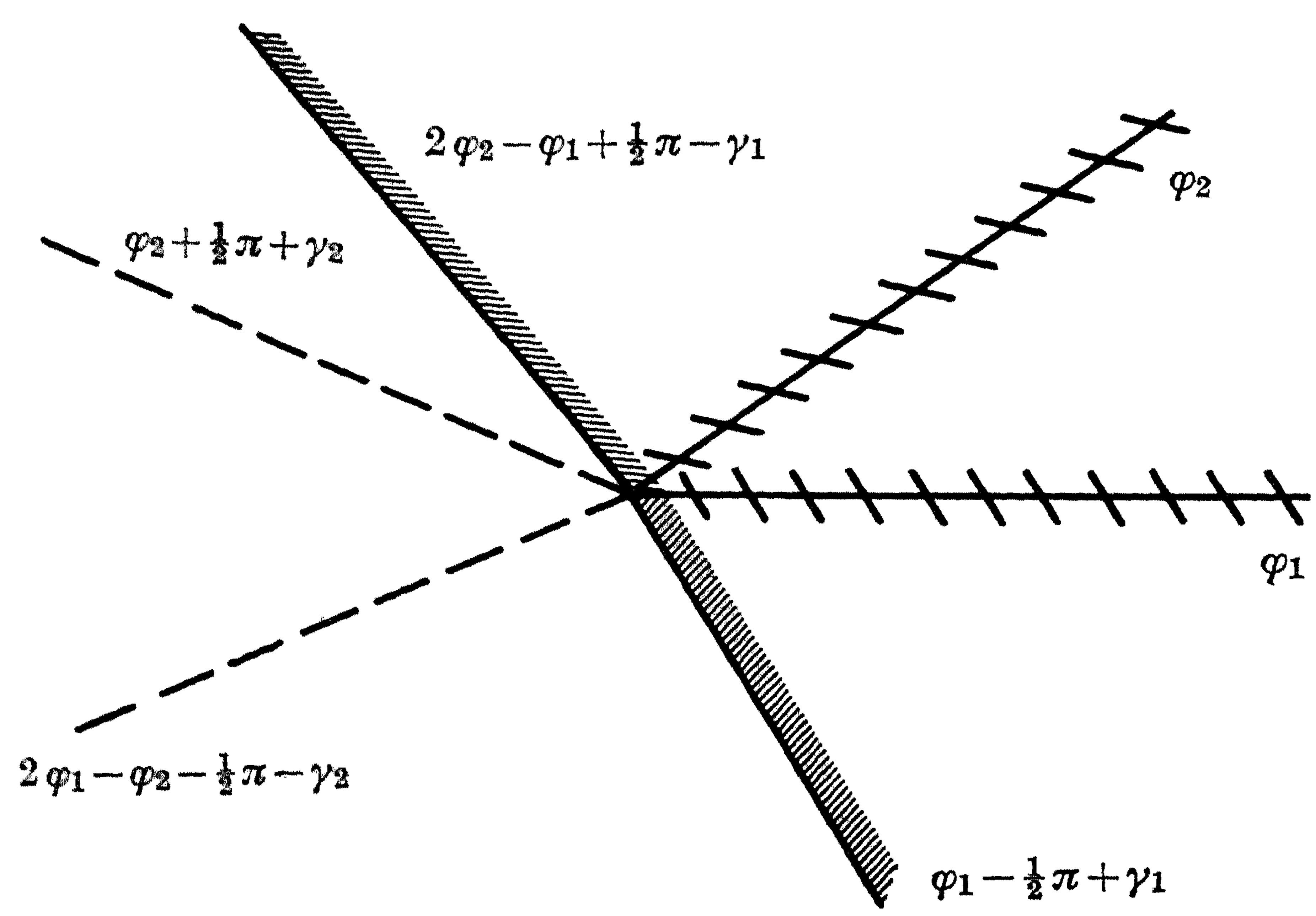

Fig. 2

In this case the solution and its analytic continuation exist at least in a halfplane viz. in the angle

(5.18) $$\varphi_1 - \tfrac{1}{2}\pi + \operatorname{Re}\gamma_1 < \varphi < \varphi_2 + \tfrac{1}{2}\pi + \operatorname{Re}\gamma_2.$$

Formula (5.11) ,still with $m = 0$, can now be written as follows

$$F(r, \varphi) = \tfrac{1}{2}\int_{L_1} e^{-r\,\mathrm{ch}\,(w - i\varphi)}\phi(w - \tfrac{1}{2}\pi i)\,dw + \tfrac{1}{2}\int_{L_2} e^{-r\,\mathrm{ch}(w - i\varphi)}\phi(w + \tfrac{1}{2}\pi i)\,dw,$$

where L_1 is the horizontal path $\operatorname{Im} w = \frac{1}{2}\pi + \varphi$ and L_2 the horizontal path $\operatorname{Im} w = -\frac{1}{2}\pi + \varphi$.

The two lines of integration can be replaced by a single one by shifting L_1 downwards and L_2 upwards. In this way we obtain

$$(5.19)\qquad F(r,\varphi)=\tfrac{1}{2}\int_{-\infty+ic}^{\infty+ic} e^{-r\operatorname{ch}(w-i\varphi)}\{\phi(w-\tfrac{1}{2}\pi i)+\phi(w+\tfrac{1}{2}\pi i)\}\,dw,$$

where c is subjected to the inequality

$$\varphi_1+\operatorname{Re}\gamma_1<c<\varphi_2+\operatorname{Re}\gamma_2.$$

We note that no poles of ϕ are passed when L_1 and L_2 are shifted in this way.

The combination $\phi(w-\frac{1}{2}\pi i)+\phi(w+\frac{1}{2}\pi i)$ is important enough to give it a special notation. We shall define

$$(5.20)\qquad H(w)=\tfrac{1}{2}\{\phi(w-\tfrac{1}{2}\pi i)+\phi(w+\tfrac{1}{2}\pi i)\}.$$

By means of (5.10) and (4.13) it can be derived that

$$(5.21)\qquad H(w)=\frac{\cos\frac{1}{2}\nu(\gamma_1-\gamma_2)\operatorname{sh}\nu(w-i\varphi_1)}{2i\operatorname{sh}\frac{1}{2}\nu(w-i\varphi_1+i\gamma_1)\operatorname{sh}\frac{1}{2}\nu(w-i\varphi_2-i\gamma_2)}\,\phi(w-\tfrac{1}{2}\pi i).$$

Hence it follows that the poles and zeros are as follows

poles
$$w=i\{\varphi_1+\gamma_1-2m\theta-n\pi\}$$
$$w=i\{\varphi_2+\gamma_2+2m\theta+n\pi\}$$
$$w=i\{\varphi_1-\gamma_2-(2m+1)\theta-n\pi\}$$
$$w=i\{\varphi_2-\gamma_1+(2m+1)\theta+n\pi\},$$

zeros
$$w=i\{\varphi_1+\gamma_1+(2m+2)\theta+(n+1)\pi\}$$
$$w=i\{\varphi_2+\gamma_2-(2m+2)\theta-(n+1)\pi\}$$
$$w=i\{\varphi_1-\gamma_2+(2m+1)\theta+(n+1)\pi\}$$
$$w=i\{\varphi_2-\gamma_1-(2m+1)\theta-(n+1)\pi\}$$
$$w=i\{\varphi_1\pm m\theta\},$$

where m, n are non-negative integers.

We note that $H(w)$ satisfies the functional relations

$$(5.22)\qquad \operatorname{sh}(w-i\gamma_j)H(i\varphi_j+w)=\operatorname{sh}(w+i\gamma_j)H(i\varphi_j-w).$$

The higher solutions with $m\geqq 1$ can be reduced in a similar way. An equivalent set of solutions is

$$(5.23)\qquad F_m(r,\varphi)=\tfrac{1}{2}\int_{-\infty+ic}^{\infty+ic}\exp(-r\operatorname{ch}(w-i\varphi))\operatorname{ch}m\nu(w-i\varphi_1)H(w)\,dw.$$

In the special case $\gamma_1=\gamma_2=0$ the solution $H(w)\equiv 1$ of (5.22) gives at once

$$F_m(r,\varphi)=K_{m\nu}(r)\cos m\nu(\varphi-\varphi_1).$$

The results of section 3 suggest that there exists a second class of solutions which are finite (or e.g. zero) at $r=0$ and which are infinite as

$r \to \infty$. In the general case such solutions can be constructed in the following way

$$(5.24) \quad F_m^*(r,\varphi) = \frac{1}{4\pi i} \int_L \exp\{r \operatorname{ch}(w - i\varphi) - \sigma m\nu (w - i\varphi_1)\} H(w)\, dw,$$

where L is a contour consisting of two parts as shown below. In each part Re w has the same sign and is sufficiently large in absolute value so that no poles of $H(w)$ are enclosed. We may assume that

$$|\mathrm{Re}\, w| > \max\{|\mathrm{Im}\,\gamma_1|, |\mathrm{Im}\,\gamma_2|\}.$$

Further $\sigma = \operatorname{sgn} \mathrm{Re}\, w$.

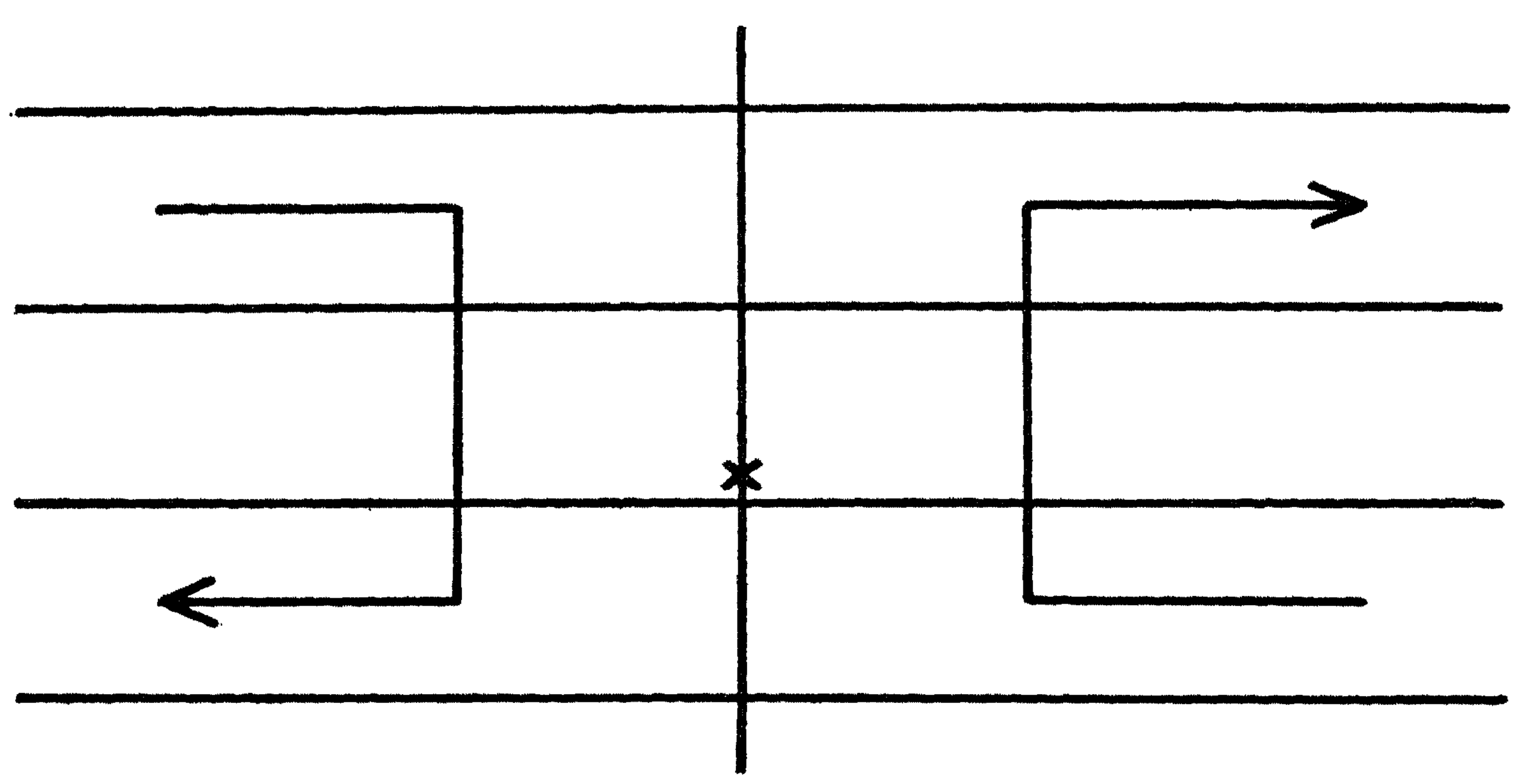

Fig. 3

It can easily be verified that (5.24) has the required properties. By way of illustration we again consider the case $\gamma_1 = \gamma_2 = 0$. In this case we have the well-known integral representation of $I_\mu(x)$ with $\mu \geqq 0$

$$(5.25) \qquad I_\mu(x) = \frac{1}{2\pi i} \int_{L^+} e^{x \operatorname{ch} w - \mu w}\, dw,$$

where L^+ is the right-hand part of the contour L (with $\varphi = 0$). Taking again $H(w) \equiv 1$ formula (5.24) gives at once

$$F_m^*(r, \varphi) = I_{m\nu}(r) \cos(m\nu(\varphi - \varphi_1)).$$

It is obvious that by (5.24) a solution is given for *all* values of φ_1, φ_2, γ_1 and γ_2 and that it can be continued for all values of φ.

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